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On the quantum theory of a laser with inhomogeneous broadening

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Abstract. The results of the semi-classical theory for the laser are compared with the results of the quantum theory given by Scully and Lamb. It is found that the spatial structure of the cavity mode leads to important corrections. For moving atoms the cavity mode is smeared, and a generalized form of the Scully–Lamb theory (for resonance between the atomic transitions and the cavity mode) leads to agreement with the semi-classical theory. The photon distribution for the case with moving atoms is obtained and compared with the results of the Scully–Lamb theory.

1. Introduction

Much of the discussion of the gas laser has been done within the framework of Lamb's (1964) semi-classical theory. This theory is capable of explaining, at least qualitatively, most observed features of laser operation. For one-mode operation the theory has been treated in the case of arbitrary intensities by Stenholm and Lamb (1969). They prove the validity of some simple results earlier conjectured by Lamb (1964) in the limit where the Doppler linewidth is much larger than the natural linewidth.

Limiting themselves to the case of stationary atoms, Scully and Lamb (1967) have incorporated the basic philosophy of the semi-classical theory into a theory which also takes into account the quantum nature of the electromagnetic field. As the case of stationary atoms can be solved exactly in the semi-classical theory, we compare the two approaches in § 2 and find that the Scully–Lamb result corresponds to the semi-classical one, provided that the spatial structure of the cavity mode is neglected. For moving atoms this is expected to be less important, as the spatial structure is smeared by the motion. We therefore use the approach of Scully and Lamb (1967) in § 3 to obtain a generalization of their equations, valid for moving atoms with an inhomogeneously broadened line due to Doppler shifts. These equations are solved under steady-state conditions in § 4 and compared with the results given by Scully and Lamb. The conclusions of the paper are summarized in § 5.

2. Comparison between quantum and semi-classical theory

The quantum theory of the laser developed by Scully and Lamb (1967) is based on the same ideas as the semi-classical theory of Lamb (1964). One assumes a highly selective optical cavity into which atoms in the upper state $|a\rangle$ of the level pair $|a\rangle$ and $|b\rangle$ are introduced at random times. The energy difference $\hbar\omega$ between $|a\rangle$ and $|b\rangle$ is assumed to be nearly in resonance with one eigenfrequency Ω of the laser cavity†. The interaction between an injected atom and the electromagnetic field in the cavity is treated exactly over a time long enough for the atom to decay spontaneously to either of two states $|c\rangle$, $|d\rangle$. This decay is treated in the Wigner–Weisskopf approximation, which makes the levels $|a\rangle$ and $|b\rangle$ appear to decay with rates γ_a and γ_b, respectively. The change in the density matrix of the radiation field is calculated and the changes $\delta\rho$ caused by a set of independent atoms are summed over a time

† The corresponding cavity eigenmode is $\sin Kz$, where $K = \Omega/c$.

Δt in order to give a coarse-grained approximation to the derivative $d\rho/dt$. The losses in the cavity are also treated in a similar way by introducing individual atoms in the lower one of two rapidly decaying (i.e. broad) levels $|\alpha\rangle$ and $|\beta\rangle$.

The basis of this approach is that atomic correlations may be neglected. In a laser appreciably above threshold it is a natural assumption to regard the electromagnetic field as a reservoir, which can absorb and emit quanta without changing significantly. This is exactly true if the field is in a coherent state, in the sense of Glauber (1963). A similar reservoir for particles is provided by the condensed bosons in the treatment of a boson system given by Bogoliubov (1947) and Hugenholtz and Pines (1959). The quantum theory determines the photon statistics and gives an expression for the average number of quanta in the lasing mode

$$\bar{n} = \frac{A}{C}\frac{A-C}{B} + \frac{A}{B}\rho_{00}. \tag{1}$$

The second term is a contribution to the lasing field from amplified spontaneous emission noise. The factor A is the mode gain and C is the loss; B is the non-linear saturation parameter. ρ_{00} is a normalization constant for the density matrix of the radiation field. When $A \gg C$ the laser is far above threshold and the last term in (1) may be neglected. Introducing the dimensionless intensity parameter

$$I = \frac{B}{A}\bar{n} = \frac{A}{C} - 1 \tag{2}$$

we can compare (2) with the results of the semi-classical theory for stationary atoms. The theory of Lamb (1964) and Stenholm and Lamb (1969) determines the laser field E from

$$E = -\frac{Q}{\epsilon_0}\frac{2}{L}\int_0^L dz\, S(z)\sin Kz \tag{3}$$

where L is the length of the cavity and Q is its quality factor (Q value). $S(z)$ is the 'out-of-phase' component of the polarization in the atomic medium. For stationary atoms $S(z)$ can be calculated exactly†:

$$S(z) = -\frac{\mathscr{P}^2 E}{\gamma_{ab}\hbar}\sin Kz\,(1+2I\sin^2 Kz)^{-1}\bar{N}. \tag{4}$$

(Here we have assumed resonance $\omega = \Omega$.) $\mathscr{P}$ is the atomic dipole matrix element coupling the states $|a\rangle$ and $|b\rangle$, and the parameter $\bar{N}$ gives the pumping rate. γ_{ab} is the average of the decay rates γ_a and γ_b. Replacing $\sin^2 Kz$ by its average value $\frac{1}{2}$, we find from (3) and (4)

$$\frac{\epsilon_0\gamma_{ab}\hbar}{Q\mathscr{P}^2\bar{N}} = \frac{1}{1+I}. \tag{5}$$

When $I = 0$ we obtain the threshold value Q_T and defining

$$A = Q_T^{-1}\Omega = \frac{\mathscr{P}^2\bar{N}\Omega}{\epsilon_0\gamma_{ab}\hbar} \tag{6}$$

$$C = Q^{-1}\Omega \tag{7}$$

we find from (5)

$$I = \frac{A}{C} - 1 \tag{8}$$

† The parameter I in equation (4) has been proved to equal the one in equation (2) (Riska and Stenholm 1969).

as in equation (2). This calculation takes the spatial structure of the cavity mode into account only in an average way and seems justified because the field is coherent over the whole cavity. This is, however, not quite correct as the stationary atoms 'see' the field at one point only and have no way of averaging the field over the cavity. The result of the semi-classical equations (3) and (4) can be evaluated exactly to give†

$$I = \frac{A}{C}\{1-(1+2I)^{-1/2}\} \tag{9}$$

from which we obtain

$$I = \left(\frac{A}{C}-\frac{1}{4}\right) - \left\{\frac{1}{2}\left(\frac{A}{C}+\frac{1}{8}\right)\right\}^{1/2} \tag{10}$$

which expanded near threshold $A \simeq C$ gives

$$I = \frac{2}{3}\left(\frac{A}{C}-1\right)+\frac{2}{27}\left(\frac{A}{C}-1\right)^2 - \dots. \tag{11}$$

The expression (11)‡ differs appreciably from (8).

In the case with moving atoms it may be expected that the atom may 'feel' the average of the field over the spatial variation in the cavity mode. The approximation called REA by Stenholm and Lamb (1969) is found to describe the case of moving atoms within a rather wide range of intensities. This gives at resonance and for a broad velocity distribution an expression for the intensity

$$\frac{\epsilon_0 \hbar K u}{\sqrt{\pi} Q \mathscr{P}^2 \bar{N}} = (1+I)^{-1/2} \tag{12}$$

where u is the width of the atomic velocity distribution. If we define A as

$$A = \sqrt{\pi}\frac{\mathscr{P}^2 \bar{N} \Omega}{\epsilon_0 K u \hbar} \tag{13}$$

we find

$$I = \left(\frac{A}{C}\right)^2 - 1. \tag{14}$$

This expression gives the same threshold condition as (8), but a steeper slope above threshold. With moving atoms the intensity is thus given by a rather simple relation (12) as compared with the result (10) for stationary atoms.

In this paper we suggest a modification of the Scully and Lamb (1967) theory by considering an atomic line which is inhomogeneously broadened by the Doppler shifts caused by atomic motion. We use the results of Scully and Lamb and find that the generalization is straightforward in the resonant case ($\omega = \Omega$), because then only one set of atoms with velocities v around zero is involved. For a detuned laser ($\omega \neq \Omega$) the two travelling-wave components of the standing wave in the cavity involve two sets of atoms, with velocities such that the Doppler shift compensates the detuning

$$\omega\frac{v}{c} = \pm|\Omega - \omega|. \tag{15}$$

† Inserting (4) in (3), we obtain an integral which can be calculated by the formula 2.458.2 of Ryshik and Gradstein (1963) and 331.21*b* of Gröbner and Hofreiter (1965). The result is given in equation (9).

‡ Inclusion of the mode structure into the quantum theory for stationary atoms has been proved to give the result (10) (Riska and Stenholm 1969).

The two sets involved complicate the use of the theory of Scully and Lamb, and we intend to discuss this case separately.

The term proportional to ρ_{00} in (1) makes it difficult to define an exact threshold in the quantum theory. If we define the threshold using only the first term in (1), we find the same condition as in the semi-classical theory, $A = C$. At threshold and below it the spontaneous field $(A/B)\rho_{00}$ dominates. In this case, however, the validity of the basic assumption that the atoms interact independently with the field can be questioned. The intensity is low and absorption of one quantum may no longer be neglected. The influence of cavity modes with energies near $\hbar\Omega$ should also be considered in this region, and might be expected to cause a shift in the threshold. Below threshold it is thus not clear whether the expression (1) can give more than a qualitative picture of the laser field. The experimental results by Sayers *et al.* (1969) do, however, confirm the existence of this term. In this paper we assume the laser to be well above threshold, and neglect the term containing ρ_{00}.

3. Equation of motion with inhomogeneous broadening

We assume that at random times atoms in the state $|a\rangle$ are injected into the cavity with a rate r_a, and, in order to describe losses, atoms in the lower non-resonant state $|\beta\rangle$ are injected with a rate r_β. The atoms are supposed to interact independently with the radiation field, which is a justified assumption well above threshold.

The Hamiltonian $\hbar H$ for the interaction between an injected atom, with the transition energy $\hbar\omega = E_a - E_b$ and a cavity mode with frequency Ω, is in the dipole approximation

$$H = \Omega a^+ a + \frac{E_b}{\hbar} + \tfrac{1}{2}\omega(1+\sigma^z) + g(a^+\sigma + a\sigma^+). \tag{16}$$

Here a^+ is the photon creation operator and σ, σ^+ are the raising and lowering operators between the states $|a\rangle$ and $|b\rangle$. The population inversion operator σ^z is defined as

$$\sigma^z = \sigma^+\sigma - \sigma\sigma^+. \tag{17}$$

The density matrix of the coupled system of the field and an atom injected at t_0 has the elements

$$\rho_{\alpha n,\beta n'} = \langle \alpha, n|\rho|\beta, n'\rangle \tag{18}$$

where $|\alpha, n\rangle$ is a state with n photons in the cavity and the atom in the state $|\alpha\rangle$ where α is one of $\{a, b, c, d\}$. The calculations of Scully and Lamb (1967) give the following equations of motion for the density matrix elements:

$$\begin{aligned}\dot\rho_{an,an'} = & -\mathrm{i}\{(n-n')\Omega - \mathrm{i}\gamma_a\}\rho_{an,an'} \\ & -\mathrm{i}\{g(n+1)^{1/2}\rho_{bn+1,an'} - g(n'+1)^{1/2}\rho_{an,bn'+1}\}\end{aligned} \tag{19a}$$

$$\begin{aligned}\dot\rho_{an,bn'+1} = & -\mathrm{i}\{(n-n')\Omega + (\omega-\Omega) - \mathrm{i}\gamma_{ab}\}\rho_{an,bn'+1} \\ & -\mathrm{i}\{g(n+1)^{1/2}\rho_{bn+1,bn'+1} - g(n'+1)^{1/2}\rho_{an,an'}\}\end{aligned} \tag{19b}$$

$$\begin{aligned}\dot\rho_{bn+1,an'} = & -\mathrm{i}\{(n-n')\Omega - (\omega-\Omega) - \mathrm{i}\gamma_{ab}\}\rho_{bn+1,an'} \\ & -\mathrm{i}\{g(n+1)^{1/2}\rho_{an,an'} - g(n'+1)^{1/2}\rho_{bn+1,bn'+1}\}\end{aligned} \tag{19c}$$

$$\begin{aligned}\dot\rho_{bn+1,bn'+1} = & -\mathrm{i}\{(n-n')\Omega - \mathrm{i}\gamma_b\}\rho_{bn+1,bn'+1} \\ & -\mathrm{i}\{g(n+1)^{1/2}\rho_{an,bn'+1} + g(n'+1)^{1/2}\rho_{bn+1,an'}\}\end{aligned} \tag{19d}$$

where $\gamma_{ab} = \frac{1}{2}(\gamma_a+\gamma_b)$, and, further,

$$\rho_{cn,cn'}(t_0+T) = \gamma_a \int_{t_0}^{t_0+T} dt' \rho_{an,an'}(t') \tag{20a}$$

$$\rho_{dn+1,dn'+1}(t_0+T) = \gamma_b \int_{t_0}^{t_0+T} dt' \rho_{bn+1,bn'+1}(t'). \tag{20b}$$

Here T is a time which is long compared with all decay times γ^{-1}. This means that we can integrate the equations (19) from t_0 to t_0+T and set

$$\begin{aligned}\rho_{an,an'}(t_0+T) &= \rho_{an,bn'+1}(t_0+T) = \rho_{bn+1,an'}(t_0+T) \\ &= \rho_{bn+1,bn'+1}(t_0+T) = 0\end{aligned} \tag{21}$$

as the atom injected at t_0 in the state $|a\rangle$ has decayed to one of the states $|c\rangle$ or $|d\rangle$ at t_0+T. The initial condition is

$$\rho_{an,an'}(t_0) = \rho_0 \tag{22a}$$

$$\rho_{an,bn'+1}(t_0) = \rho_{bn+1,an'}(t_0) = \rho_{bn+1,bn'+1}(t_0) = 0 \tag{22b}$$

where the initial value ρ_0 will be specified later.

The integration of equations (19) leads to a set of linear equations for the quantities

$$\sigma_{11}(n, n') = \int_{t_0}^{t_0+T} dt' \rho_{an,an'}(t') \tag{23a}$$

$$\sigma_{12}(n, n') = \int_{t_0}^{t_0+T} dt' \rho_{an,bn'+1}(t') \tag{23b}$$

$$\sigma_{21}(n, n') = \int_{t_0}^{t_0+T} dt' \rho_{bn+1,an'}(t') \tag{23c}$$

$$\sigma_{22}(n, n') = \int_{t_0}^{t_0+T} dt' \rho_{bn+1,bn'+1}(t') \tag{23d}$$

which in matrix form is

$$\begin{bmatrix} \Omega(n-n')-i\gamma_a & -g(n'+1)^{1/2} & g(n+1)^{1/2} & 0 \\ -g(n'+1)^{1/2} & \Omega(n-n')+\Delta-i\gamma_{ab} & 0 & g(n+1)^{1/2} \\ g(n+1)^{1/2} & 0 & \Omega(n-n')-\Delta-i\gamma_{ab} & -g(n'+1)^{1/2} \\ 0 & g(n+1)^{1/2} & -g(n'+1)^{1/2} & \Omega(n-n')-i\gamma_b \end{bmatrix} \times \begin{bmatrix} \sigma_{11}(n, n') \\ \sigma_{12}(n, n') \\ \sigma_{21}(n, n') \\ \sigma_{22}(n, n') \end{bmatrix} = \begin{bmatrix} -i\rho_0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{24}$$

where $\Delta = \omega - \Omega$.

The solution for $\sigma_{11}(n, n')$ and $\sigma_{22}(n, n')$ is then

$$\begin{bmatrix} \sigma_{11}(n, n') \\ \sigma_{22}(n, n') \end{bmatrix} = -\frac{i\rho_0}{D(n, n')} \times \begin{bmatrix} \Omega(n-n')-i\gamma_b-g^2(\Sigma^2-\Delta^2)^{-1}[\{(n+1)+(n'+1)\}\Sigma-(n-n')\Delta] \\ -2g^2\{(n+1)(n'+1)\}^{1/2}(\Sigma^2-\Delta^2)^{-1}\Sigma \end{bmatrix} \tag{25}$$

where

$$\Sigma = \omega(n-n') - \mathrm{i}\gamma_{ab} \tag{26}$$

and

$$D(n, n') = \Sigma^2 + \tfrac{1}{4}(\gamma_a - \gamma_b)^2 + g^2(\Sigma^2 - \Delta^2)^{-1} \times [(n-n')^2 g^2 - \mathrm{i}(n-n')(\gamma_a - \gamma_b)\Delta - 2\{(n+1)+(n'+1)\}\Sigma^2]. \tag{27}$$

We now deviate from the treatment of Scully and Lamb and assume that the atoms have an inhomogeneous distribution of their resonance frequency ω. We take the stationary atoms to be in resonance with the field and assume atomic motion to cause a Doppler shift in the frequency

$$\omega = \Omega + \frac{\Omega}{c} v = \Omega + Kv. \tag{28}$$

Instead of introducing one single atom into the cavity, we introduce an ensemble of independent atoms with a velocity distribution

$$W(v) = \frac{1}{u\sqrt{\pi}} \exp\left(-\frac{v^2}{u_2}\right) \tag{29}$$

where we assume the velocity width u to be very much larger than γ_{ab}/K (the Doppler limit). As the injected atoms at $t = t_0$ are independent of the field, we have

$$\rho_{an,an'}(t)_0 = \rho_0 = W(v)\rho_{nn'}(t_0). \tag{30}$$

We calculate the change in the density matrix of the field $\rho_{nn'}$ due to this ensemble of atoms injected at $t = t_0$. We have

$$\delta\rho_{nn'} = \rho_{nn'}(t_0+T) - \rho_{nn'}(t_0) = \sum_{\alpha} \left\{\int_{-\infty}^{\infty} \mathrm{d}v\, \rho_{\alpha n,\alpha n'}(t_0+T)\right\} - \rho_{nn'}(t_0). \tag{31}$$

Here we have contracted with respect to the atomic indices $\alpha = a, b, c, d$ and summed over all velocities v. From (20) and (21) it follows that

$$\rho_{nn'}(t_0+T) = \int_{-\infty}^{\infty} \mathrm{d}v\, \{\gamma_a \sigma_{11}(n, n') + \gamma_b \sigma_{22}(n-1, n'-1)\} \tag{32}$$

which is a slight generalization of the result of Scully and Lamb. The number of atoms introduced during an interval Δt is $r_a \Delta t$, so that the total change of the field during this time is

$$\Delta\rho_{nn'} = r_a \Delta t\, \delta\rho_{nn'}. \tag{33}$$

From (33) we obtain an expression for the time derivative of the density matrix due to atomic gain

$$\begin{aligned}\left(\frac{\mathrm{d}\rho_{nn'}}{\mathrm{d}t}\right)_{\text{gain}} &= \frac{\Delta\rho_{nn'}}{\Delta t} \\ &= -\mathrm{i}\gamma_a r_a \rho_{nn'}(t) \int_{-\infty}^{\infty} \mathrm{d}v\, W(v)\tilde{D}^{-1}(n, n')[(\Sigma^2 - K^2v^2)\{\Omega(n-n') - \mathrm{i}\gamma_b\} \\ &\quad - g^2\{(n+1)+(n'+1)\}\Sigma + g^2(n-n')Kv] \\ &\quad + \mathrm{i}\gamma_b r_a \rho_{n-1,n'-1}(t) \int_{-\infty}^{\infty} \mathrm{d}v\, W(v)\tilde{D}^{-1}(n-1, n'-1) 2g^2\, \Sigma\, (nn')^{1/2} \\ &\quad - r_a \rho_{nn'}(t) \int_{-\infty}^{\infty} \mathrm{d}v\, W(v)\end{aligned} \tag{34}$$

where

$$\tilde{D}(n, n') = (\Sigma^2 - K^2v^2)D(n, n') \tag{35}$$

and the v dependence of ω is introduced from equation (28).

The loss mechanism is treated in a similar way. The only difference is that the atoms are now injected in the lower state $|\beta\rangle$ at a rate r_β. The resulting change in the density matrix due to the loss mechanism is

$$\begin{aligned}\left(\frac{\mathrm{d}\rho_{nn'}}{\mathrm{d}t}\right)_{\text{loss}} &= \mathrm{i}\gamma_\alpha r_\beta \rho_{n+1,n'+1}(t) \int_{-\infty}^{\infty} \mathrm{d}v\, W(v)\tilde{D}^{-1}(n, n') \\ &\quad \times 2g^2 \Sigma\, \{(n+1)(n'+1)\}^{1/2} - \mathrm{i}\gamma_\beta r_\beta \rho_{nn'}(t) \\ &\quad \times \int_{-\infty}^{\infty} \mathrm{d}v\, W(v)\tilde{D}^{-1}(n-1, n'-1)[(\Sigma^2 - K^2v^2)\{\Omega(n-n') - \mathrm{i}\gamma_\alpha\} \\ &\quad - g^2(n+n')\Sigma - g^2(n-n')Kv] - r_\beta \rho_{nn'}(t).\end{aligned} \tag{36}$$

Here $\tilde{D}(n, n')$ is given by (27) and (35), where a and b have been replaced by α and β. The loss is treated as a linear process, and linearizing equation (36) we obtain

$$\left(\frac{\mathrm{d}\rho}{\mathrm{d}t}\right)_{\text{loss}} = C\{(n+1)(n'+1)\}^{1/2}\rho_{n+1,n'+1} - \tfrac{1}{2}C(n+n')\rho_{nn'} \tag{37}$$

where

$$C = 2g^2 r_\beta \gamma_{\alpha\beta} \int_{-\infty}^{\infty} \frac{\mathrm{d}v\, W(v)}{\gamma_\beta(\gamma_{\alpha\beta}{}^2 + K^2v^2)}. \tag{38}$$

The pair $|\alpha\rangle$, $|\beta\rangle$ represents a loss system well, when a quantum absorbed in the transition $|\beta\rangle \rightarrow |\alpha\rangle$ is not re-emitted before the state $|\alpha\rangle$ has decayed. Therefore $|\alpha\rangle$ and $|\beta\rangle$ should be broad levels. Assuming that $\gamma_\alpha, \gamma_\beta \gg Ku$, we can take the Lorentzian in (38) outside the integral, and write

$$C \simeq \frac{2g^2 r_\beta}{\gamma_\beta \gamma_{\alpha\beta}}. \tag{39}$$

We can now write the equation of motion for the density matrix of the electromagnetic field, including both amplification and loss terms combining (34) and (37):

$$\begin{aligned}\frac{\mathrm{d}\rho_{nn'}}{\mathrm{d}t} &= -\mathrm{i}\gamma_a r_a \rho_{nn'}(t) \int_{-\infty}^{\infty} \mathrm{d}v\, W(v)\tilde{D}^{-1}(n, n') \\ &\quad \times [(\Sigma^2 - K^2v^2)\{\Omega(n-n') - \mathrm{i}\gamma_b\} - g^2\{(n+1)+(n'+1)\}\Sigma + g^2(n-n')Kv] \\ &\quad + \mathrm{i}\gamma_b r_a \rho_{n-1,n'-1}(t) \int_{-\infty}^{\infty} \mathrm{d}v\, W(v)\tilde{D}^{-1}(n-1, n'-1) 2g^2 \Sigma\, (nn')^{1/2} \\ &\quad - r_a \rho_{nn'}(t) + C\{(n+1)(n'+1)\}^{1/2}\rho_{n+1,n'+1}(t) \\ &\quad - \tfrac{1}{2}C(n+n')\rho_{nn'}(t).\end{aligned} \tag{40}$$

4. Steady-state conditions

The diagonal part of equation (40) will contain velocity integrals of the type

$$\begin{aligned}I(n) &= \int_{-\infty}^{\infty} \mathrm{d}v\, W(v) \frac{2g^2\gamma_b\gamma_{ab}n}{\gamma_a\gamma_b(\gamma_{ab}{}^2 + K^2v^2) + 4g^2\gamma_{ab}{}^2 n} \\ &= \frac{2g^2 n}{\gamma_a K u\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{d}x\, \frac{\gamma_{ab}\exp\{-x^2/(Ku)^2\}}{x^2 + \gamma_{ab}{}^2\{1 + (4g^2/\gamma_a\gamma_b)n\}}.\end{aligned} \tag{41}$$

When

$$Ku \gg \gamma_{ab}\left(1+\frac{4g^2}{\gamma_a\gamma_b}n\right)^{1/2} \tag{42}$$

we can perform the integration in (41) approximatively:

$$I(n) \simeq \frac{2\sqrt{\pi}g^2}{Ku\gamma_a}\left(1+\frac{4g^2}{\gamma_a\gamma_b}n\right)^{1/2}. \tag{43}$$

Defining†

$$A = \frac{2\sqrt{\pi}g^2 r_a}{\gamma_a Ku}, \qquad B = \frac{4g^2}{\gamma_a\gamma_b}A \tag{44}$$

we find the diagonal part of (40) to be

$$\begin{aligned}\dot{\rho}_{nn} = &-A(n+1)\left\{1+\frac{B}{A}(n+1)\right\}^{-1/2}\rho_{nn}-Cn\rho_{nn}\\ &+An\left\{1+\frac{B}{A}n\right\}^{-1/2}\rho_{n-1,n-1}+C(n+1)\rho_{n+1,n+1}.\end{aligned} \tag{45}$$

The steady-state solution $\dot{\rho}_{nn} = 0$ of this equation is obtained when

$$\rho_{n+1,n+1} = \frac{A}{C}\left\{1+\frac{B}{A}(n+1)\right\}^{-1/2}\rho_{nn} \tag{46}$$

and the solution is obviously

$$\rho_{nn} = \left(\frac{A}{C}\right)^n \rho_{00}\prod_{\nu=0}^{n}\left(1+\frac{B}{A}\nu\right)^{-1/2} \tag{47}$$

where ρ_{00} is chosen so that

$$\sum_n \rho_{nn} = 1. \tag{48}$$

The result (47) differs from that of Scully and Lamb only by the presence of the square root in the denominator, but this leads to some significant differences in the photon distribution. The threshold condition is still $A = C$. The amplification factor of Scully and Lamb is

$$A_{\mathrm{SL}} = 2r_a\frac{g^2}{\gamma_a\gamma_{ab}} \tag{49}$$

and we have

$$A = \sqrt{\pi}\frac{\gamma_{ab}}{Ku}A_{\mathrm{SL}} \tag{50}$$

exactly as in the semi-classical theory. The amplification factor A for moving atoms (equation (13)) is $\sqrt{\pi}(\gamma_{ab}/Ku)$ times the one for stationary atoms (equation (6)). This factor derives from the fact that, with moving atoms, only the fraction γ_{ab}/Ku of the atoms is able to sustain the laser oscillations.

† From Scully and Lamb (1967) it follows that the coupling constant g is given by $g^2 = \mathscr{P}^2\Omega/2\hbar\epsilon_0 V$, where V is the volume of the cavity. Introducing this into (44) and setting $r_a/N = \bar{N}$, we regain equation (13).

Above threshold, $A > C$, ρ_{nn} increases with n up to a value $\bar{n}$, determined by the equation

$$\frac{A}{C} = \left(1 + \frac{B}{A}\bar{n}\right)^{1/2}; \tag{51}$$

for $n > \bar{n}$ the value of ρ_{nn} decreases monotonically. ρ_{nn} is the probability of finding n photons in the cavity. This probability distribution has a maximum at

$$\bar{n} = \frac{A}{B}\left\{\left(\frac{A}{C}\right)^2 - 1\right\} \tag{52}$$

in contrast to the result of Scully and Lamb

$$\bar{n}_{\mathrm{SL}} = \frac{A}{B}\left(\frac{A}{C} - 1\right). \tag{53}$$

In terms of the dimensionless intensity parameter I given by equation (2), we find that our result (52) exactly corresponds to the semi-classical result (14) derived by Stenholm and Lamb.

The three expressions for I given in (2), (10) and (14) are compared in figure 1 as a function of the relative excitation parameter A/C. The photon distribution (47) is compared with that of Scully and Lamb in figure 2.

The width of the photon distribution can be obtained in the way used by Scully (1965). From (47) we have

$$\rho_{\bar{n}+k,\bar{n}+k} = \left(\frac{A}{C}\right)^k \prod_{\nu=1}^{k}\left\{1 + (n+\nu)\left(\frac{B}{A}\right)\right\}^{-1/2} \rho_{\bar{n},\bar{n}}. \tag{54}$$

The half-width is obtained if we put $\rho_{\bar{n}+k,\bar{n}+k} = \frac{1}{2}\rho_{\bar{n},\bar{n}}$, giving

$$\tfrac{1}{4} = \prod_{\nu=1}^{k}\left\{\left(\frac{C}{A}\right)^2\left(1 + \bar{n}\frac{B}{A}\right) + \nu\left(\frac{C}{A}\right)^2\frac{B}{A}\right\}^{-1}. \tag{55}$$

Using (52) and assuming that C^2B/A^3 is a small parameter, we obtain

$$\tfrac{1}{4} \simeq \prod_{\nu=1}^{k}\left(1 - \nu\frac{C^2B}{A^3}\right) \simeq 1 - \frac{C^2B}{A^3}\sum_{\nu=1}^{k}\nu \simeq 1 - \frac{k^2}{2}\frac{C^2B}{A^3}. \tag{56}$$

From (56) we obtain the variance of the photon distribution

$$\sigma^2 = k^2 = \tfrac{3}{2}\bar{n}\frac{(A/C)^2}{(A/C)^2 - 1}. \tag{57}$$

The corresponding result of the Scully–Lamb theory is

$$\sigma_{\mathrm{SL}}{}^2 = \bar{n}_{\mathrm{SL}}\frac{A/C}{A/C - 1}. \tag{58}$$

If we fix A/C in such a way that $\bar{n}$ in (52) is equal to $\bar{n}_{\mathrm{SL}}$ in (53), we find from (57) and (58)

$$\frac{\sigma^2}{\sigma_{\mathrm{SL}}{}^2} = \frac{3}{2}\frac{A/C}{A/C - 1} > 1 \tag{59}$$

showing that we obtain a larger width σ^2 than $\sigma_{\mathrm{SL}}{}^2$.

In order to test the validity of the approximate calculation for the half-width, we calculate $\sigma^2/\bar{n}$ exactly using a computer. In figure 3 the results are compared with the approximate expressions (57) and (58). The exact half-widths are larger than the approximate ones. The asymptotic values for large A/C do not seem to be given correctly by the approximative method, and the approximation is not sensitive enough to show the relation between our results and those of Scully and Lamb.

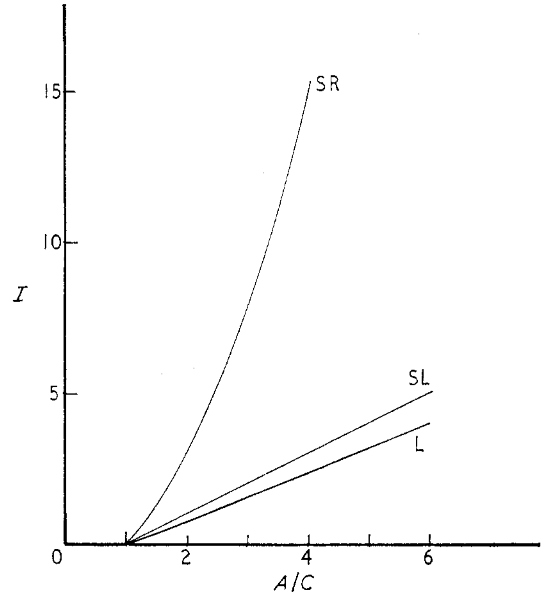

Figure 1. The laser intensity I for the case of stationary atoms in the semi-classical theory (L) and quantum theory (SL) compared with the case of moving atoms (SR), which gives identical results in the semi-classical and quantum theory. The corresponding expressions are equations (10), (2) and (14).

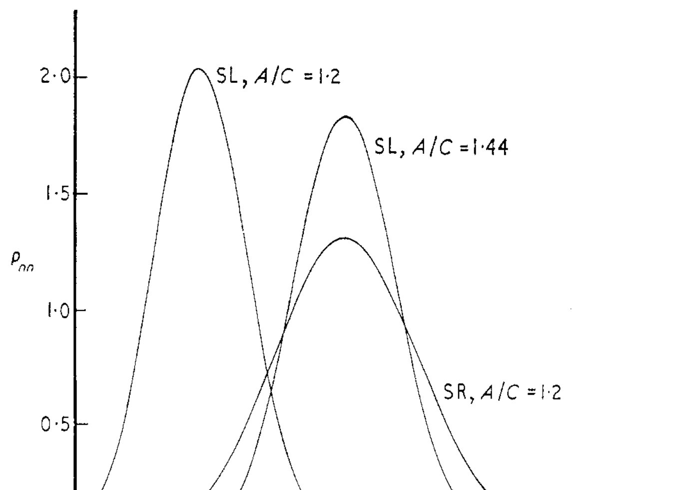

Figure 2. The photon distribution, equation (47), for moving atoms (SR) compared with the distribution found by Scully and Lamb for stationary atoms (SL) for the parameter values $A/C = 1{\cdot}2$, $1{\cdot}44$ with $B/A = 0{\cdot}005$.

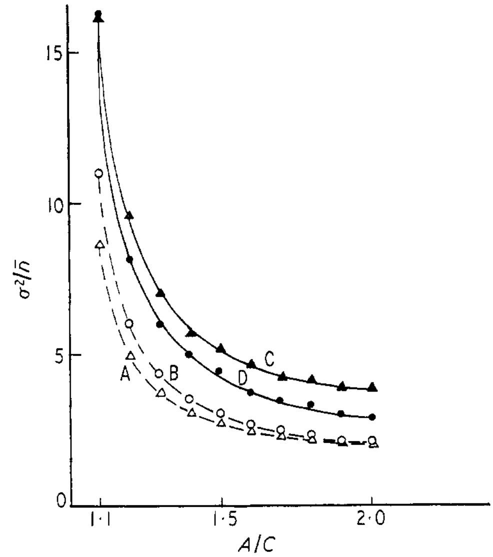

Figure 3. The approximative values of $\sigma^2/\bar{n}$ compared with the exact ones: A, approximate curve for moving atoms (SR); B, approximate curve for stationary atoms (SL); C, exact curve for moving atoms (SR); D, exact curve for stationary atoms (SL). The parameter B/A is chosen to be 0·005.

5. Conclusions

The results of the theory of Scully and Lamb (1967) are found to agree with the semi-classical theory of Lamb (1964) only if the detailed structure of the electromagnetic mode is neglected in the latter. The exact result (10) is more complicated than the Scully–Lamb result. This is based on the assumption that each atom interacts equally strongly with the field. When the atoms are stationary this procedure seems to be inadequate as some atoms are situated exactly at the nodes of the field and therefore can undergo no transitions in the semi-classical picture. This effect of the mode structure gives rise to the difference between the expressions (8) and (10). As soon as the atoms move they see an average over the field and the mode structure becomes less important. For exact resonance between the atomic transitions and the cavity, $\omega = \Omega$, only atoms with their Doppler-shifted transition frequencies within an atomic linewidth γ_{ab} from Ω interact with the field, and the approach of Scully and Lamb may be used. By drawing heavily on their work we obtain an expression for the photon distribution, which gives a threshold identical with the one derived by Stenholm and Lamb (1969) in the semi-classical theory. This fact verifies our statement that the influence of the mode structure (which has been neglected both by Scully and Lamb and us) is less important in the case of moving atoms than in the one of stationary atoms. For a detuned laser ($\omega \neq \Omega$) the standing wave of the electromagnetic field involves two sets of atoms. In that case the treatment of Scully and Lamb becomes more difficult, and we plan to discuss this problem later.

The main results of the present theory are the following. Above threshold the intensity (14) rises more steeply than in the case of stationary atoms, equations (2) and (10) (see figure 1). For the same value of A/C we thus find a larger photon number $\bar{n}$ (see figure 2 for $A/C = 1{\cdot}2$). The line is broader than that of Scully and Lamb also when A/C is adjusted, so as to give $\bar{n} = \bar{n}_{\mathrm{SL}}$ (see figure 2, the Scully and Lamb result for $A/C = 1{\cdot}44$). The approximative method used by Scully and Lamb to obtain the half-width is found to be inadequate to distinguish between the results of Scully and Lamb and our results. For large A/C the variance of Scully and Lamb is equal to the variance of a Poisson distribution, which is the distribution for an electromagnetic field in a coherent state (Glauber 1963). The asymptotic value of (57) is larger, but figure 3 suggests that the exact asymptotic limits are different.

This paper emphasizes the importance of atomic motion in smearing the mode structure in a laser. An expression for the photon distribution is given, and compared with the expression given by Scully and Lamb. It is also pointed out that both expressions should be used only well above threshold, as the derivation rests on the assumption that the number of photons constituting the field is large.

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